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ABERDEEN PROVING GROUND, MARYLAND

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by

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METHODS OF TABULATING BALLISTIC FUNCTIONS BASED ON THE SQUARE LAW OF DRAG

ABSTRACT

The solution of the equations of motion of a projectile subject to the Newtonian Law of Resistance is expressible in terms of quadratures. By applying certain transformations these quadratures are now greatly simplified, and the functions leading to the desired information pertaining to the trajectory are quickly and accurately calculated. There is, then, given a method of determination of the basic parameter k which leads to very good results, as shown by numerous comparative examples.

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LIST OF SYMBOLS AND FORMULAS

$C = m/id^2$	Ballistic coefficient
$G = v\rho_0 K_D$	Resistance function
$H = \rho/\rho_0$	Relative air density
K_D	Drag coefficient
$T = T(\beta, b)$	Time function
$X = X(\beta, b)$	Range function
$Y = Y(\beta, b)$	Altitude function
b	$\xi(\theta_0) + g/(2k\dot{x}_0^2)$, quantity associated with the square law
c	$c(\xi) = \cos\theta(\xi)$
g	Gravitational acceleration
k	$\rho K_D/C$, quantity associated with the square law
s	$s(\xi) = \sin\theta(\xi)$
t	Time
v	Speed of projectile
v_0, v_s, v_ω	Speed of projectile at point of departure, summit, point of fall, respectively
x	Abscissa (horizontal) in system $O(x, y)$ of reference describing the motion of the projectile
y	Ordinate (vertical) in system $O(x, y)$

$\beta = (b - \xi)/b$

$\gamma = \beta^{1/2}$

$\eta = -\xi$

θ	Inclination of trajectory
$\theta(\xi)$	Inverse function of $\xi(\theta)$

$\xi = \xi(\theta) = \int_0^\theta \sec^3\theta \, d\theta$

ρ	Air density at altitude y
ω	Angle of fall

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INTRODUCTION

The application of the Newtonian Law of Resistance to ballistic investigations has been found to give exact results only in rather special cases, but does lead to good approximations in many more cases of practical interest. There are certain projectiles whose velocities are limited to relatively small ranges, and whose resistance it is difficult to determine accurately. As pointed out by L. S. Dederick and F. V. Reno[1], S. J. Zaroodny and D. N. Brooks[2], and others, it is for projectiles such as these that the square law of drag may profitably be employed - provided one assumes the product of air density and drag coefficient to be constant over the whole trajectory.

While extensive tables of exterior ballistics based on the square law of drag - such as, for example, the tables of Otto and Lardillon - have existed for a long time, it was, nevertheless, felt that an accurate tabulation of the square law ballistic functions themselves would not only be useful, but would also fill a gap hitherto existing in this field. It is for these reasons that the job was begun, early in 1945, and carried out on the Aberdeen IBM Relay Calculators, under the direction of I. J. Schoenberg.

There are thus now available six tables, which, it is hoped will greatly facilitate the use of the square law in its region of applicability[3]. The functions tabulated are

1. $\xi(\theta) = \int_0^{\theta} \sec^3 \theta \, d\,\theta.$

This table is the same as the one published in Cranz[4], but improved by the weeding out of numerous errors. In this table, which gives $\xi(\theta)$ for $\theta = 0^\circ 0' (1') 87^\circ 0'$, the number of decimal places exhibited drops from seven for low θ to four for large values of θ.

2. $s(\xi) = \sin \theta(\xi)$, to nine decimals, for $\xi = 0.00\ (0.01)\ 50.00$.

   Here $\theta(\xi)$ is the inverse function of $\xi(\theta)$.

3. $c(\xi) = \cos \theta(\xi)$, to eight decimals, for the same range of ξ.

4. $X(\beta, b) = \int_1^{\beta} c(\xi)\, d\beta/\beta$.

---

[1] Dederick - Reno, "Solution of Differential Equations of Motion of a Projectile with Newtonian Resistance in an Atmosphere of Variable Density", BRL No. 243, 1942.

[2] Zaroodny - Brooks, "Charts for the Exterior Ballistics of Mortar Fire Based on the Square Law of Drag", BRL No. 661, 1948.

[3] The Ballistic Research Laboratories have a limited supply of these tables for distribution, upon request, to those agencies requiring them.

[4] C. Cranz, "Lehrbuch der Ballistik", Vol. 1, Berlin, 1925.

5. $Y(\beta, b) = \int_1^{\beta} s(\xi)\, d\beta/\beta.$

6. $T(\beta, b) = \int_1^{\beta} c(\xi)\, d\beta/\beta^{1/2}.$

These last three functions X, Y, and T, have been tabulated to eight decimal places, for b = 0.1 (0.1) 2.0, β = 0.00 (0.02) 3.00, but with the upper limits of these ranges restricted by $\beta\,(b-1) \leq 2$, as shown in Fig. 2, p. 16. To facilitate interpolations all tables also show the differences up to the third order.

Most computations were conducted to eight or more decimal places, and checked by differencing, in the case of X, Y, T, with respect to b as well as β. A detailed description of the construction of each table will be found in the later sections. There is, finally, suggested a method of obtaining the basic quantity k from the ballistic coefficient C which leads to very good results for the range and time of flight, as shown by various examples.

THE DIFFERENTIAL EQUATIONS OF THE TRAJECTORY

The motion of the center of mass of a projectile will in the following be referred to a system of Cartesian coordinates O (x,y) having its origin O at the point of departure, the x-axis being horizontal and positive in the direction of fire, and the y-axis being vertical and positive upwards; see Fig. 1.

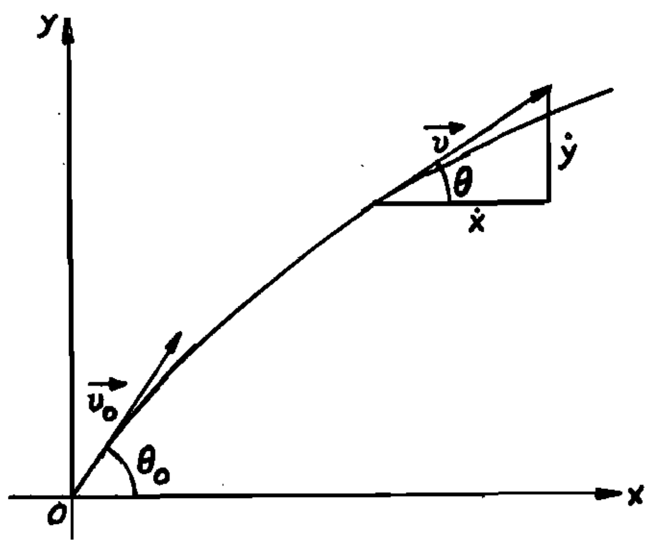

Fig. 1

In such a system of coordinates the equations of motion may be written in the well-known form

$$\ddot{x} = -E\dot{x}$$
$$\ddot{y} = -E\dot{y} - g, \tag{1}$$

where the quantities $\dot{x}$, $\ddot{x}$, etc. occurring in these equations have the following meanings:

$\dot{x} = dx/dt$, t being the time,

$\ddot{x} = d^2x/dt^2$, $\dot{y} = dy/dt$,

g is the gravitational acceleration, and

$$E = G\,H/C. \tag{2}$$

Here

$$H = \rho/\rho_o = e^{-ay}$$

is the ratio of air density at altitude y to that at sea level,

$$G = v\rho_o K_D$$

is the resistance function, K_D the drag coefficient, v the speed of the projectile, and C the ballistic coefficient. Making use of these formulas for G, H, and C, E becomes

$$E = kv,$$

where

$$k = \rho K_D/C. \tag{3}$$

Since K_D is dimensionless, k itself has the dimension of a reciprocal length. While k, by (3), is actually dependent on altitude y and Mach number M, it will here be assumed constant along the whole trajectory.

If, under this assumption, the angle θ of inclination of the trajectory - rather than the time t - is taken as the independent variable, the equations of motion may be replaced by the equivalent system[1]

$$\begin{aligned} gd(v\cos\theta) &= kv^3\,d\theta \\ gdx &= -v^2 d\theta \\ gdy &= -v^2\tan\theta\,d\theta \\ gdt &= -v\sec\theta\,d\theta. \end{aligned} \tag{4}$$

[1] This is shown, e.g., in Cranz, "Lehrbuch der Ballistik, p. 110.

The "principal equation" (4.1)[1] integrates into

$$g(v \cos \theta)^{-2} = -2k\xi(\theta) + 2kb, \tag{5}$$

with

$$\xi(\theta) = \int_0^\theta \sec^3 \theta \, d\theta, \tag{6}$$

and 2kb being the constant of integration. For the initial conditions: at $t = t_o$, $x = x_o$, $y = y_o$, $v = v_o$, $\theta = \theta_o$, b obviously becomes

$$b = g/(2k\dot{x}_o^2) + \xi_o \tag{7}$$

with

$$\xi_o = \xi(\theta_o),$$

$$\dot{x}_o = v_o \cos \theta_o.$$

Since at the summit S of the trajectory $\theta = 0$, $v = v_s$, b may also be characterized by

$$b = g/(2kv_s^2). \tag{8}$$

Solving eq. (5) for v^2 leads to

$$v^2 = g \sec^2 \theta/2k\,(b - \xi(\theta)),$$

whence

$$\begin{aligned} 2\,k\,dx &= -\sec^2 \theta \, d\theta/(b - \xi) \\ 2\,k\,dy &= -\tan \theta \sec^2 \theta \, d\theta/(b - \xi) \\ (2kg)^{1/2} dt &= -\sec^2 \theta \, d\theta/(b - \xi)^{1/2}. \end{aligned} \tag{9}$$

TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS

While the solution of the problem is thus reduced to quadratures, some further simplications may be achieved by introducing the inverse function $\theta(\xi)$ of $\xi(\theta)$. Let us put

$$c(\xi) = \cos \theta(\xi), \quad s(\xi) = \sin \theta(\xi) \tag{10}$$

By virtue of $d\theta = c^3(\xi)d\xi$ above equations (9) take on the form

$$\begin{aligned} 2\,k\,dx &= -c(\xi)\, d\xi/(b - \xi) \\ 2\,k\,dy &= -s(\xi)\, d\xi/(b - \xi) \\ (2kg)^{1/2} dt &= -c(\xi)d\xi/(b - \xi)^{1/2} \end{aligned} \tag{11}$$

[1] By equation (4.1) is meant the first equation of set (4).

If we now define a new variable β by means of

$$b - \xi = b\beta,$$

i.e. put

$$\xi = b(1 - \beta), \tag{12}$$

equations (11) further simplify to

$$\begin{aligned} 2\,k\,dx &= c(\xi)\,d\beta/\beta \\ 2\,k\,dy &= s(\xi)\,d\beta/\beta \\ 2\,kv_s dt &= c(\xi)\,d\beta/\beta^{1/2}. \end{aligned} \tag{13}$$

By virtue of equations (5) and (8) the variable β turns out to be

$$\beta = v_s^2 \sec^2 \theta / v^2. \tag{14}$$

Therefore, as θ decreases from θ_o to $\theta_s = 0$, ξ decreases from ξ_o to 0, and β increases from

$$\beta_o = (v_s \sec \theta_o / v_o)^2 = 1 - \xi_o / b$$

to $\beta_s = 1$; as the projectile moves down the descending branch β is increased further. In terms of the parameter β the trajectory is now represented by

$$\begin{aligned} 2k(x - x_o) &= X(\beta, b) - X(\beta_o, b) \\ 2k(y - y_o) &= Y(\beta, b) - Y(\beta_o, b) \\ 2kv_s(t - t_o) &= T(\beta, b) - T(\beta_o, b) \end{aligned} \tag{15}$$

where

$$\begin{aligned} X(\beta, b) &= \int_1^{\beta} c(\xi)\,d\beta/\beta \\ Y(\beta, b) &= \int_1^{\beta} s(\xi)\,d\beta/\beta \\ T(\beta, b) &= \int_1^{\beta} c(\xi)\,d\beta/\beta^{1/2} \end{aligned} \tag{16}$$

and

$$\xi = b(1 - \beta).$$

The integrations, described in the following sections, were actually carried out not on the integrals (16), but on related integrals, obtained as follows: First it is advantageous to replace ξ by

$$\eta = -\xi = b(\beta - 1). \tag{17}$$

Since $\theta(\xi)$ is an odd function, $s(\xi)$ is also odd, while $c(\xi)$ is even:

$$s(\xi) = -s(\eta); \quad c(\xi) = c(\eta).$$

Further we may write

$$X(\beta, b) = \overline{X}(\beta, b) + c(b) \ln \beta, \tag{18}$$

where

$$\overline{X}(\beta, b) = \int_1^{\beta} [c(\eta) - c(b)] \, d\beta/\beta, \tag{19}$$

and

$$Y(\beta, b) = \overline{Y}(\beta, b) + s(b) \ln \beta \tag{20}$$

with

$$\overline{Y}(\beta, b) = \int_1^{\beta} [-s(\eta) - s(b)] \, d\beta/\beta. \tag{21}$$

The transformation (17) changes the time integral (16.3) into

$$T(\beta, b) = \int_1^{\beta} \beta^{1/2} \, c(\eta) d\beta/\beta.$$

It is convenient here to introduce

$$\gamma = \beta^{1/2},$$

so that

$$T(\beta, b) = \int_1^{\gamma} S(\gamma, b) d\gamma \tag{22}$$

with

$$S(\gamma, b) = 2c(\eta), \quad \eta = b(\gamma^2 - 1), \quad \gamma^2 = \beta.$$

The problem is thus reduced to the calculation of $\overline{X}$, $\overline{Y}$, and T.

As the basis for all necessary integrations there was used the formula

$$\Delta'_o = \int_{\beta_o}^{\beta_1} f(\beta) d\beta = (h/2)(f_o + f_1) + (h^2/10)(\dot{f}_o - \dot{f}_1) + (h^3/120)(\ddot{f}_o + \ddot{f}_1) + R, \tag{23}$$

where dots ($\dot{}$) denote differentiation with respect to β. This formula may be derived as follows: Expanding Δ'_o in Taylor series about β_o, $\beta_1 = \beta_o + h$, respectively, we have

$$\Delta'_o = \sum_{i=1}^{\infty} (h^i/i!) \, f_o^{(i-1)},$$

and

$$\Delta_0' = \sum_{i=1}^{\infty} \left[(-1)^{i-1} h^i / i!\right] f_1^{(i-1)}.$$

By addition, then,

$$\Delta_0' = (h/2)\ (f_0 + f_1) + (h^2/4)\ (\dot{f}_0 - \dot{f}_1) + (h^3/12)\ (\ddot{f}_0 + \ddot{f}_1) + 0(h^4).$$

If we now split up the h^2 term into two parts one of which is $(h^2/10) \times (\dot{f}_0 - \dot{f}_1)$, and expand the other part, again by a Taylor series, we get the formula (23) with

$$R = 0(h^4).$$

Having obtained the X, Y, and T for certain equidistant key arguments β a subtabulation was performed by means of second order osculation, in the following manner: In order to calculate the value of a function $g(t)$ tabulated for equidistant $t = t_i$ by means of osculatory interpolation we put

$$t = t_0 + \tau h, \quad h = t_1 - t_0$$

$$g(t_0 + \tau h) = \phi(\tau) \tag{24}$$

and get[1]

$$\phi(\tau) = D_0 + D_1 \tau + \dots + D_5 \tau^5 \tag{25}$$

where the D_i are the following functions of $\phi_j = \phi(j)$, $j = 0, 1$, $\phi_j' = d\phi_j/d\tau, \phi_j''$:

$$D_0 = \phi_0$$

$$D_1 = \phi_0' = h g_0',$$

$$D_2 = (1/2)\phi_0'' = (h^2/2) g_0''.$$

$$D_3 = -10\phi_0 - 6\phi_0' + 10\phi_1 - 4\phi_1' + (1/2)\phi_1'' - (3/2)\phi_0''$$

$$D_4 = 15\phi_0 + 8\phi_0' - 15\phi_1 + 7\phi_1' - (1/2)\phi_1'' + (3/2)\phi_0''$$

$$D_5 = -6\phi_0 - 3\phi_0' + 6\phi_1 - 3\phi_1' + (1/2)\phi_1'' - (1/2)\phi_0''.$$

Rearranging the terms in formula (25) so as to exhibit the factors of ϕ_0, ϕ_1, etc., there results

$$\phi(\tau) = \phi_0 P_0(\tau) + \phi_0' P_1(\tau) + \phi_0'' P_2(\tau) + \phi_1 Q_0(\tau) + \phi_1' Q_1(\tau) + \phi_1'' Q_2(\tau) \tag{26}$$

[1] For a derivation of this formula, see, for example, LOTKIN, M., "Inversion on the Eniac Using Osculatory Interpolation," BRL 632, 1947.

with

$$P_0(\tau) = 1 - 10\tau^3 + 15\tau^4 - 6\tau^5$$

$$P_1(\tau) = \tau(1 - 6\tau^2 + 8\tau^3 - 3\tau^4)$$

$$P_2(\tau) = (1/2)\ \tau^2\ (1 - \tau)^3$$

$$Q_0(\tau) = 1 - P_0(\tau)$$

$$Q_1(\tau) = -\tau^3(4 - 7\tau + 3\tau^2)$$

$$Q_2(\tau) = (1/2)\tau^3\ (1 - \tau)^2.$$

For subtabulation to $\Delta\tau = 0.1$ these polynomials P_i, Q_i assume the values shown in Table I.

TABLE I. INTERPOLATION COEFFICIENTS

	$P_0(\tau)$	$P_1(\tau)$	$P_2(\tau)$	$Q_0(\tau)$	$Q_1(\tau)$	$Q_2(\tau)$
0.1	0.991440	0.094770	0.003645	0.008560	-0.003330	0.000405
0.2	0.942080	0.163840	0.010240	0.057920	-0.021760	0.002560
0.3	0.836920	0.195510	0.015435	0.163080	-0.058590	0.006615
0.4	0.682560	0.190080	0.017280	0.317440	-0.107520	0.011520
0.5	0.500000	0.156250	0.015625	0.500000	-0.156250	0.015625
0.6	0.317440	0.107520	0.011520	0.682560	-0.190080	0.017280
0.7	0.163080	0.058590	0.006615	0.836920	-0.195510	0.015435
0.8	0.057920	0.021760	0.002560	0.942080	-0.163840	0.010240
0.9	0.008560	0.003330	0.000405	0.991440	-0.094770	0.003645

COMPUTATION OF THE FUNCTION $\xi(\theta)$

Since $\xi(\theta)$, as defined by (6), is an odd function of θ, it is positive along the ascending branch, vanishes at the summit, and becomes negative along the descending branch of the trajectory. It may be written explicitly as

$$\xi(\theta) = (1/2)\left[\tan\theta\sec\theta + \text{log nat}\tan(\pi/4 + \theta/2)\right]. \qquad (27)$$

$\xi(\theta)$ has been tabulated in Cranz, for every minute of arc, from $\theta = 0^\circ$ to $\theta = 87^\circ 0'$, to 8 or less significant figures. Upon differencing this table up to differences of third order two types of errors showed up, due, obviously, either to misprints or small miscalculations. A listing of these values of $\xi(\theta)$ that were found to be in error by 5 or more in the last decimal shown is given in Table II. The errors, when not due to misprint, were eliminated by recomputation on the basis of eq. (27), in the following manner: Making use of the nine place WPA Table of trigonometric functions for each 0.0001 radians of argument, we write,

TABLE II. ERRORS IN CRANZ'S TABLE OF $\xi(\theta)$

θ		CRANZ		CORRECT VALUE		ERROR
3°	13'	0.056	2500	0.056	2300	+ 200
7	28	0.131	3552	0.131	4352	- 800
14	14	0.256	3625	0.256	3525	+ 100
15	17	0.276	6189	0.276	6199	- 10
17	47	0.326	1715	0.326	1615	+ 100
18	25	0.339	0377	0.339	0337	+ 40
20	57	0.392	0202	0.392	0212	- 10
21	25	0.402	0083	0.402	0683	- 600
21	37	0.406	4046	0.406	4036	+ 10
23	55	0.457	6223	0.457	6233	- 10
25	16	0.488	0669	0.488	9669	- 9000
25	34	0.496	0831	0.496	0731	+ 100
27	54	0.553	2731	0.553	2631	+ 100
29	21	0.590	6981	0.590	6891	+ 90
35	42	0.777	3334	0.776	3334	+ 10000
58	27	2.198	352	2.188	352	+10000X
63	06	2.893	664	2.893	654	+ 10X
66	39	3.700	597	3.710	597	-10000X
69	24	4.643	374	4.633	374	+10000X
73	55	7.238	203	7.238	293	- 90X
81	03	21.681	00	21.681	08	- 8XX
83	00	34.811	31	34.811	36	- 5XX
84	16	51.348	13	51.348	02	+ 11XX
85	23	78.532	3	78.534	1	- 18XXX
86	59	182.103	0	182.103	7	- 7XXX

$$\theta = \theta_o + n, \quad \psi = \pi/4 + \theta/2 = \psi_o + p,$$

where θ_o, ψ_o represent the first four decimals only of θ and ψ, expressed in radians to, say, 9 decimals. The values of $\sin\theta$, $\cos\theta$ are then obtained with sufficient accuracy from

$$\sin\theta = (1 - (n^2/2)) \sin\theta_o + n\cos\theta_o$$

$$\cos\theta = (1 - (n^2/2)) \cos\theta_o - n\sin\theta_o$$

since the error incurred in cutting off the series after the terms shown here is $R \leq n^2/6!$, $n \leq 1.10^{-4}$.

Having thus found $\sin\theta$, $\cos\theta$, $\sin\psi$, $\cos\psi$, $\tan\theta\sec\theta$ and $\tan\psi$ were easily computed. Putting, as above,

$$\tan\psi = (\tan\psi)_o + q,$$

where $(\tan\psi)_o$ represents the first four decimals only of $\tan\psi$, and using the notation $r = q/(\tan\psi)_o$, so that

$$\tan\psi = (\tan\psi)_o\,(1 + r),$$

$\ln\tan\psi$ was calculated sufficiently accurately from

$$\ln\tan\psi = \ln\,(\tan\psi)_o + r - r^2/2$$

with the nine place values of $\ln(\tan\psi)_o$ extracted from the WPA Table.

THE TABLES OF $s(\xi)$ and $c(\xi)$

If we consider θ as a function of ξ, eq. (27) may be expressed as

$$4\xi = 2s(\xi)/c^2(\xi) + \ln\tan^2[\pi/4 + \theta(\xi)/2],$$

so that

$$\xi = Q(s) = (1/4)\left[(1 - s)^{-1} - (1 + s)^{-1} + \ln(1 + s) - \ln(1 + s)\right]. \tag{28}$$

From a table of $Q(s)$ the function $s(\xi)$ may thus be computed by inverse interpolation.

To construct an eight place table of $Q(s)$ by means of eq. (28) for the arguments $s = 0.0001$ (0.0001) 0.9960 it was necessary first to calculate reciprocals $w = 1/u$ for $u = 0.0040$ (0.0001) 0.9999 to eight decimals. Using Barlow's Table to get a first approximation w_o of w, correct to five decimals, a correction dw_o was obtained in the usual manner: If $Z(u, w) = 0$, and $w = w_o + dw_o$, then

$$dw_o = -Z(u, w_o)/Z_w(u, w_o),\; Z_w = \partial Z/\partial w. \tag{29}$$

Since here $\mathbf{Z}(u, w) = u - w^{-1}$, $dw_o = w_o(1 - uw_o)$. The values of the natural logarithms needed were again extracted from the WPA Table.

Having made this table an approximate value s_o of s was then found by inversion for each $\zeta = 0.1\ (0.1)\ 2.0\ (0.05)\ 50.00$ and improved by

$$ds_o = - \mathbf{Z}(\zeta, s_o)/\mathbf{Z}_s(\zeta, s_o);$$

in this case $\mathbf{Z}(\zeta, s) = \zeta - Q(s)$, so that

$$ds_o = \left[\zeta - Q(s_o)\right]/Q'(s_o),$$

and

$$Q'(s) = (1/4)\left\{(1 - s)^{-1}\left[1 + (1 - s)^{-1}\right] + (1 + s)^{-1}\left[1 + (1 + s)^{-1}\right]\right\}$$

This process was continued until s had been determined accurately to nine places, for the key arguments ζ listed above. Next a subtabulation to tenths was performed, and the $s(\zeta)$ obtained for $\zeta = 0.00\ (0.01)\ 2.00$ by means of the doubly osculating cubic

$$\phi(\tau) = E_o + E_1\tau + E_2\tau^2 + E_3\tau^3$$

with $E_o = \phi_o$

$$E_1 = \phi_o'$$

$$E_2 = -3\phi_o + 3\phi_1 - 2\phi_o' - \phi_1'$$

$$E_3 = 2\phi_o - 2\phi_1 + \phi_o' + \phi_1',$$

giving an eight place accuracy in the interpolated values, with a possibly wrong round-off in the ninth place. The $s(\zeta)$ table was then checked by forming the advancing differences up to the fifth order.

Later calculations required knowledge of derivatives $s' = dx/d\zeta$, s'', and s'''. By Eq. (10)

$$s' = c^4(\zeta) = (1 - s^2)^2.$$

Therefore,

$$s'' = -4s(1 - s^2)^3,$$

and

$$s''' = 4(1 - s^2)^4\left[6 - 7(1 - s^2)\right].$$

These derivatives, then, could be computed easily once $s = s(\zeta)$ was known.

The determination of the function $c(\zeta)$, too, was based on that of

$s(\xi)$, since

$$c(\xi) = \left[1 - s^2(\xi)\right]^{1/2}$$

The eight place values of the square roots were obtained by second order interpolation

$$\phi(\tau) = \phi_0 + (1/2)\ (\Delta_0 + \Delta_1)\ \tau + (1/2)\Delta_1''\tau^2$$

in an eight place table of roots for arguments 0.0001 (0.0001) 0.9999.

The derivatives of c, also needed later, were obtained by repeated differentiation of

$$c^2(\xi) + s^2(\xi) = 1;$$

expressed in terms of the derivatives s' and s'' they are:

$$c' = -ss'/c$$

$$c'' = -(c'^2 + s'^2 + ss'')/c$$

$$c''' = -\left[3(c'c'' + s's'') + ss'''\right]/c$$

or

$$c''' = sc^7\ (28c^2 - 15),$$

where the last relationship may be obtained by differentiation from

$$c' = -sc^3,\ s' = c^4.$$

THE FUNCTION $X(\beta, b)$

From Eq. (16.1) it is obvious that $X \gtrless 0$ for $\beta \gtrless 1$, for all values of b. As already pointed out previously, the actual calculation of X was based on Eqs. (18) and (19). The function $\overline{X}(\beta, b)$ was first computed for b = 0.1 (0.1) 2.0 and β = 0.0 (0.1) 3.0, with the upper ranges of b, β restricted by $\beta\,(b - 1) \leq 2.0$, as shown in Figure 2.

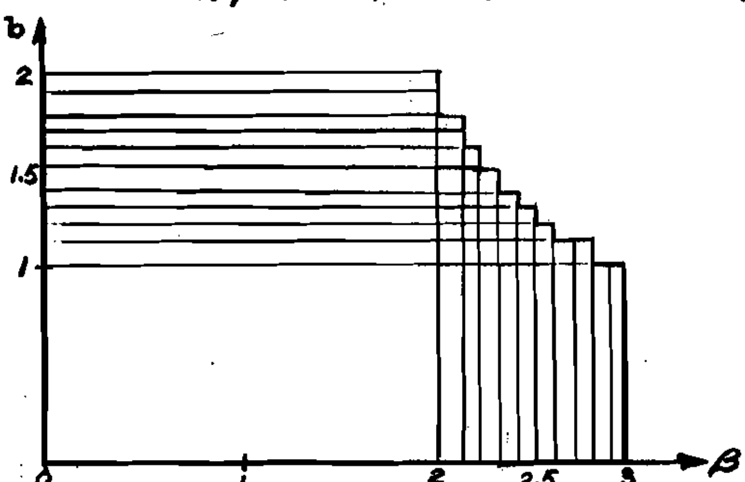

Fig. 2

As is easily seen in this region also $\bar{X} \lesseqgtr 0$ for $\beta \lesseqgtr 1$.

In using Eq. (23) for the integration of $\bar{X}$ we put $f(\beta) = A(\beta) + B(\beta)$ with $A(\beta) = c(\eta)/\beta$, $B(\beta) = -c(b)/\beta$, thus considering b as a fixed parameter. The derivatives of f occurring in Eq. (23) were then obtained from

$$\dot{A} = (bc' - A)/\beta$$

$$\ddot{A} = (b^2 c'' - 2\dot{A})/\beta$$

$$\dot{B} = -B/\beta$$

$$\ddot{B} = -2\dot{B}/\beta$$

The values of

$$\partial^i f(o)/\partial\beta^i = \partial^i A(o)/\partial\beta^i + \partial^i B(o)/\partial\beta^i, \; i = o, 1, 2,$$

which were needed to get started, were found by L'Hospital's rule:

$$f_o = -bc'(b)$$

$$\dot{f}_o = (b^2/2)\, c''(b)$$

$$\ddot{f}_o = -(b^3/3) c'''(b).$$

By determining the increment Δ^{i+1} on the basis of Eq. (23), and taking advantage of the fact that

$$\bar{X}(1, b) = 0 \text{ for all } b,$$

the function $\bar{X}(\beta, b)$ was calculated to ten places, for the range of β and b shown above.

The values of the increments $\Delta\bar{X}$ thus computed were then checked by comparing $\Delta_i^{i+2} = \Delta_i^{i+1} + \Delta_i^{i+2}$ with Δ'^{j+1}_j obtained by applying Eq. (23) to $h' = 2h$; this check showed the $\Delta\bar{X}$'s to be accurate in most cases to eight decimals. The integrations were checked further by differencing the $\bar{X}$'s listed for all **b**, but fixed β.

These $\bar{X}$'s were next subtabulated to fifths in the argument β, by means of formula (26), which becomes, in this case

$$\bar{X}(\beta + \tau h) = \bar{X}_o P_o(\tau) + h(A_o + B_o)\, P_1(\tau) + h^2 (\dot{A}_o + \dot{B}_o)\, P_2(\tau)$$

$$+ \bar{X}_1\, Q_o(\tau) + h(A_1 + B_1)\, Q_1(\tau) + h^2(A_1 + B_1)\, Q_2(\tau).$$

With these values of $\bar{X}$ on hand, it was, finally, an easy matter to get the X's themselves.

THE FUNCTION $Y(\beta, b)$

As regards the function $Y(\beta, b)$ it is again apparent from Eq. (16.2) that $Y < o$ for all b, β. Now for $o < \beta < 1$ $\xi = b(1 - \beta) < b$, so that $s(\xi) < s(b)$, and, therefore, $\bar{Y} > o$. For $\beta > 1$, on the other hand, it follows from $Y = \bar{Y} + s(b) \ln \beta$, since $Y < o$ and $s(b) \ln \beta > o$, that necessarily $\bar{Y} < o$.

The region of integration was the same as used for the X. In Eq. (23) we had to put here, however,

$$f(\beta) = M(\beta) + N(\beta)$$

with

$$M(\beta) = - s(\eta)/\beta \qquad N(\beta) = - s(b)/\beta$$

Consequently,

$$\dot{M} = - (bs' + M)/\beta \qquad \ddot{M} = - (b^2 s'' + 2\dot{M})/\beta$$

$$\dot{N} = - N/\beta \qquad \ddot{N} = - 2\dot{N}/\beta$$

For $\beta = 0$

$$f_o = - bs'(b)$$

$$\dot{f}_o = (b^2/2)\, s''(b)$$

$$\ddot{f}_o = - (b^3/3)\, s'''(b).$$

The determination of the $\Delta \bar{Y}$, their checking, and the subsequent subtabulation were carried out in the same manner as previously described for the function $\bar{X}$.

THE FUNCTION $T(\beta, b)$

The time integral (16.3) behaves similarly as the range integral X, so that $T \gtrless 0$ for $\beta \lessgtr 1$. As mentioned above, it was calculated by means of Eq. (22). To make use of the integration formula (23) it was first necessary to determine the quantities

$$h_i = \beta_i^{1/2} - \beta_{i-1}^{1/2}, \quad i = 1(1)\,30;$$

this was done to ten decimals. The derivatives of S with respect to γ are

$$S' = 4\, b\, \beta^{1/2}\, c'(\eta)$$

$$S'' = 4\, b\, c'(\eta) + 8\, b^2 \beta\, c''(\eta).$$

Checking of the values of ΔT was performed in the same manner as was

done for the $\Delta \bar{X}$ and $\Delta \bar{Y}$, and showed the ΔT, too, to be accurate generally to eight decimals.

Since the $h = \Delta \gamma$ depend on β, the subtabulation of T to fifths was a rather tedious job. It was, obviously, necessary to get for each of the four values of $\gamma_j = [\beta + (2j/100)]^{1/2}$, $\beta = 0.0\ (0.1)\ 3.0$, $j = 1, 2, 3, 4$, the value of $\tau_j = (\gamma_j - \gamma)/h$, then compute the interpolational polynomials $P_k(\tau_j)$, $Q_k(\tau_j)$, $k = 0, 1, 2$, and, finally, to use formula (26) in the form

$$T(\gamma + \tau_j h) = T_o P_o(\tau_j) + hS_o P_1(\tau_j) + h^2 S_o' P_2(\tau_j) +$$
$$T_1 Q_o(\tau_j) + hS_1 Q_1(\tau_j) + h^2 S_1' Q_2(\tau_j).$$

TRAJECTORIES FOR LARGE θ_o AND C

The tables described in the previous sections permit the calculation of trajectories to a certain limited extent. For such angles θ_o, or ballistic coefficients C that cause b, Eq. (7) to fall outside of the range of the tables the trajectories may be easily obtained by means of three simple quadratures, as follows: Let us transform the integrals (16) by means of Eq. (17) to

$$\begin{aligned} X_1(\eta, b) &= \int_0^{\eta} c(\eta)\, d\eta/(\eta + b) \\ -Y_1(\eta, b) &= -\int_0^{\eta} s(\eta)\, d\eta/(\eta + b) \end{aligned} \qquad (30)$$

and

$$b^{-1/2} T_1(\eta, b) = b^{-1/2} \int_0^{\eta} c(\eta)\, d\eta/(\eta + b)^{1/2},$$

whence

$$\begin{aligned} 2k(x - x_o) &= X_1(\eta, b) - X_1(\eta_o, b) \\ -2k(y - y_o) &= Y_1(\eta, b) - Y_1(\eta_o, b) \\ 2kv_s\, b^{1/2}(t - t_o) &= T_1(\eta, b) - T_1(\eta_o, b). \end{aligned} \qquad (31)$$

The tables of the functions $c(\eta)$ and $s(\eta)$ have been computed sufficiently far so as to make the calculation of the integrals (30) an easy matter.

In order to get the range x_ω by means of formulae (31) one first determines $\eta_o = b(\beta_o - 1) < 0$ and then computes $Y_1(\eta_o, b)$. Starting again from $\eta = 0$ the quantity $\eta_\omega > 0$ is next calculated by integrating to that value $\eta = \eta_\omega$ that will make $Y_1(\eta_\omega, b) = Y_1(\eta_o, b)$. Finally the range x_ω is obtained by computing $X_1(\eta_o, b)$ and $X_1(\eta_\omega, b)$, and applying formula (31.1).

USE OF THE TABLES

The proper use of the tables described in the previous sections will now be illustrated by means of a few examples; for the sake of simplicity let $x_o = y_o = t_o = o$.

1. Given k, θ_o, v_o. To find the angle of fall ω, range x_ω, time of flight t_ω, and striking velocity v_ω.

(a) Compute $b = \xi(\theta_o) + g/(2k\, v_o^2 \cos^2 \theta_o)$.

(b) Get $\beta_o = 1 - \xi_o/b$.

(c) By interpolation find $\beta = \beta_\omega$ so that

$$0 = 2ky_\omega = Y(\beta_\omega, b) - Y(\beta_o, b).$$

(d) By interpolation find ω so that

$$\xi(\omega) = b(1 - \beta_\omega).$$

(e) Obtain the horizontal range x_ω from

$$2kx_\omega = X(\beta_\omega, b) - X(\beta_o, b). \tag{31}$$

(f) Calculate the time of flight t_ω from

$$(2kg/b)^{1/2}\, t_\omega = T(\beta_\omega, b) - T(\beta_o, b).$$

(g) Finally, the striking velocity is found from

$$v_\omega^2 = g \sec^2\omega/2kb\,\beta_\omega.$$

2. Given v_o, θ_o, x_ω. This case can be reduced to the previous one by assuming a trial value of k, and then computing the quantities b, β_o, and β_ω. This is done until k satisfies equation (31).

The quantities t_ω and v_ω are then obtained as above.

3. Given k, v_o, x_ω. Assume a trial value of θ_o, compute b, β_o, β_ω. The correct θ_o must again satisfy equation (31).

DETERMINATION OF k

The calculation of that value of k from the data C, θ_o, v_o which will produce the same range and time of flight as the one obtained by numerical integration is a difficult job. It is, however, not quite so complicated to get a k that will give a range good enough to be used as an approximation. Such a value of k may be based on initial and summital velocity v_o and v_s, respectively, as follows:

(a) Compute $k_o = G(v_o)/v_o C$.

(b) Get $b_o = g/(2k_o \dot{x}_o^2) + \xi(\theta_o)$.

(c) Determine $v_{os} = (g/2k_o b_o)^{1/2}$.

(d) Calculate $v_m = (1/2)\ (v_o + v_{os})$.

(e) Put $k = G(v_m)/v_m C$.

(f) Compute $b = g/(2k\dot{x}_o^2) + \xi(\theta_o)$, and proceed as in example 1.

This procedure for getting k is obviously justified since, as seen in Fig. 3, the motion of the projectile along its trajectory is duplicated along the K_D - curve as a motion, starting at O, in the direction of decreasing v, having a point of reversal of direction at S', which is located slightly past the summit S, and terminating at T.

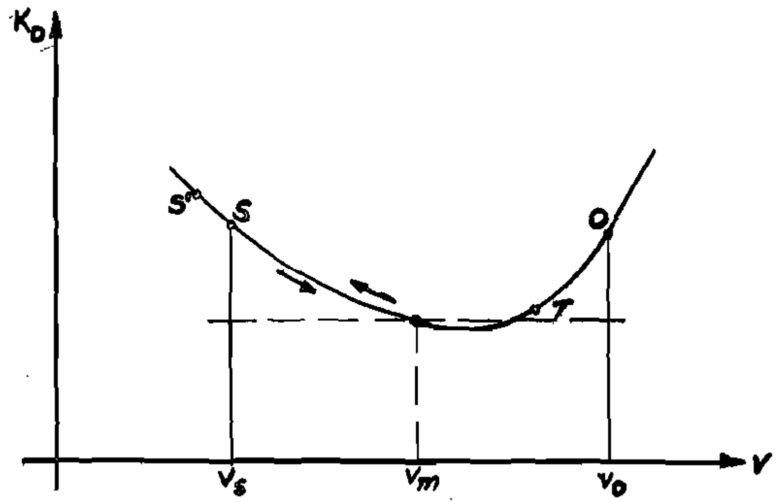

Fig. 3

In Table III there are compared summital velocities v_s, ranges x_ω and times of flight t_ω obtained by means of the square law with those calculated on the basis of the resistance functions G_i. In cases 3, 4, 5 and 6 the values of v_s, t_ω, and x_ω, for the square law, were secured by means of formulae (30).

As the table shows, the agreement between the trajectories is remarkable, the discrepancy amounting, at worst, to about 2% for the cases exhibited.

TABLE III. COMPARISON OF TRAJECTORIES

	θ_o (degs)	v_o (ft/sec)	C	v_s (ft/sec) SQ.L.	v_s (ft/sec) G_i	t_ω (sec) SQ.L.	t_ω (sec) G_i	x_ω (yards) SQ.L.	x_ω (yards) G_i
1	30	900	$C_1 = 0.5$	457	459	22.3	22.2	3462	3484
2	40	800	$C_1 = 0.6$	388	391	26.4	26.4	3475	3502
3	40	412	$C_1 = 0.978$	281	281	15.8	15.8	1473	1479
4	68	747	$C_1 = 1.65$	227	231	39.2	39.5	2972	3035
5	65	900	$C_1 = 3.7$	340	345	49.2	48.6	5469	5591
6*	$75°8'.8$	639.9	$C_5 = 1.55$	145	145	36.4	36.6	1762	1773

* This example is a particular case pertaining to the 105mm Howitzer for which a special G function is available.

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